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A Note on Property T for C^*-algebras

Firuz Kamalov

Mathematics Department
Canadian University of Dubai
Dubai, UAE

Ho-Hon Leung

Mathematics Department
Canadian University of Dubai
Dubai, UAE

Abstract

In this note we introduce an alternative definition of Property T for C^*-algebras based on the spectrum of a C^*-algebra. We show that a group G has Property T if and only if $C_r^*(G)$ has Property T. In addition, we introduce and investigate relative Property T for C^*-algebras.

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1 Introduction

There exist several equivalent statements of Property T for groups. Various authors have tried to extend these definitions from groups to C^*-algebras [1, 4, 6]. The original definition of Property T for C^*-algebras introduced by Bekka in [1] has deservedly received the most attention [2, 3, 5, 7]. However, a slightly stronger definition of Property T given by Leung and Ng seems to be more fruitful. In this note we introduce an alternative definition of Property T for C^*-algebras using the spectrum of a C^*-algebra. Our definition is inspired by a similar definition for groups. In Section 2, we give the definition of Property T and show that a discrete group G has Property T if and only if its reduced group C^*-algebra $C_r^*(G)$ has Property T.

In Section 3, we define relative Property T. We concentrate our analysis on Property T relative to the set of finite dimensional Hilbert bimodules.

2 Property T

Let G be a locally compact group and $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of G. Then we know that G has Property T if and only if every finite dimensional irreducible representation of G is isolated in $\widehat{G}$. Let A be a C^*-algebra and $\widehat{A}$ be the set equivalence classes of irreducible representations of A. Recall that $\widehat{A}$ is endowed with the pull-back topology from Prim (A). We introduce the following definition of Property T for C^*-algebras.

Definition 1. *Let A be a unital C^*-algebra. We say that A has Property T if every finite dimensional irreducible representation of A is isolated in $\widehat{A}$.*

The above definition is similar to the definition proposed by Pavlov and Troitsky in [6]. However, we believe that our definition is more appropriate in at least one important case. In particular, $C(X)$ has Property T if and only if X is finite. This is a natural result as amenability and Property T traditionally only coincide in finite dimensional cases. In general, any finite dimensional C^*-algebra has Property T.

Let G be a discrete group. Let $C^*(G)$ denote the group C^*-algebra and $C_r^*(G)$ denote the reduced group C^*-algebra.

Theorem 2. *Let G be a discrete group. Then the following statements are equivalent:*

1. *G has Property T.*
2. *$C^*(G)$ has Property T.*
3. *$C_r^*(G)$ has Property T.*

Proof. The equivalence of (1) and (2) is well known. We will show the equivalence of (1) and (3). Suppose that $C_r^*(G)$ does not have Property T. Then there exists a finite dimensional irreducible representation π_0 of $C_r^*(G)$ and a net $\{\pi_i\}$ in $\widehat{C_r^*(G)}$ such that $\pi_i \to \pi_0$. Since G is embedded in $C_r^*(G)$ we can take π_0 and $\{\pi_i\}$ as representations of G. We will show that π_0 is in the closure of the net $\{\pi_i\}$ in $\widehat{G}$. Let $s \in G$ be in the intersection $\bigcap_i \ker \pi_i$. Since s is also an element of $C_r^*(G)$, then $s \in \ker \pi_0$. Therefore, $\bigcap_i \ker \pi_i \subseteq \ker \pi_0$. It follows that G does not have Property T.

Conversely, suppose that G does not have Property T. Then the trivial representation of G, 1_G, is not isolated in $\widehat{G}$. Let $\{\pi_i\}$ be a net in $\widehat{G}$ such that $\pi_i \to 1_G$. Let λ_G be the regular representation of G. Then $\lambda_G \otimes \pi_i \to \lambda_G$. Since $\lambda_G \otimes \pi_i$ is equivalent to a multiple of λ_G we can extend $\lambda_G \otimes \pi_i$ to a representation of $C_r^*(G)$. It follows $\lambda_G \otimes \pi_i \to \lambda_G$ as representations of $C_r^*(G)$. Therefore, $C_r^*(G)$ does not have Property T.

□

Unfortunately, it remains an open question whether our definition of Property T is equivalent to that of Bekka. We only remark that in the redundant case when a C^*-algebra A does not have a tracial state A has Property T by either definition.

3 Relative Property T

In this section we would like to reconsider Bekka's definition in a more liberal sense. Recall that a Hilbert bimodule over a C^*-algebra A is a Hilbert space $\mathcal{H}$ carrying a pair of commuting representations, one of A and one of its opposite algebra. A sequence of unit vectors $\{\xi_i\}$ in $\mathcal{H}$ is called almost central vectors if $\| a\xi_i - \xi_i a \| \to 0$ for all $a \in A$.

Definition 3. *Let $\mathcal{R}$ be a set of Hilbert bimodules of A. We say that A has Property $(T, \mathcal{R})$ if for every bimodule $\mathcal{H}$ in $\mathcal{R}$ that has a sequence of almost central vectors there is a nonzero central vector in $\mathcal{H}$.*

Note that if $\mathcal{R}$ is the set of all Hilbert bimodules of A, then we obtain Bekka's original definition of Property T. We are particularly interested in the case when $\mathcal{R}$ is the set of all finite dimensional Hilbert bimodules of A. We need the following lemma for our main result.

Lemma 4. *Let π be a finite dimensional representation of A and ρ be an irreducible representation of A such that $\ker \pi \subseteq \ker \rho$. Then ρ is a subrepresentation of π.*

Proof. Since π is finite dimensional it decomposes as a finite direct sum of irreducible representations. Then $\ker \pi = \bigcap_{i=1}^n Q_i$, where $Q_i \in \mathrm{Prim}(G)$. Since $\ker \rho$ is prime and $\bigcap Q_i \subseteq \ker \rho$, then $Q_j \subseteq \ker \rho$ for some j. But Q_j is the kernel of a finite dimensional irreducible representation of A so Q_j is a maximal ideal. Therefore, $Q_j = \ker \rho$. Note that finite dimensional irreducible representations of a C^*-algebra are equivalent if and only if they have the same kernel. It follows that ρ is equivalent to a subrepresentation of π. □

Theorem 5. *Let G be a discrete group and let $\mathcal{F}$ be the set of finite dimensional Hilbert bimodules of $C^*(G)$. Then $C^*(G)$ has Property $(T, \mathcal{F})$.*

Proof. Let $\mathcal{H}$ be a finite dimensional Hilbert bimodule of $C^*(G)$ with a sequence of almost central vectors $\{\xi_i\}$. Define a representation π of G on $\mathcal{H}$ by $\pi(s)\xi = s\xi s^{-1}$ for all $s \in G$ and $\xi \in \mathcal{H}$. Then $\pi(s)\xi \to 0$ for all $s \in G$. So the representation 1_G is weakly contained in π. Since π is finite dimensional, then by the above lemma 1_G is a subrepresentation of π. Then there exists a nonzero vector $\xi_0 \in \mathcal{H}$ such that $\pi(s)\xi_0 = \xi_0$ for all $s \in G$. It follows that $s\xi_0 = \xi_0 s$ for all $s \in G$. Using linearity and continuity we get that $a\xi_0 = \xi_0 a$ for all $C^*(G)$. □

The next result is another example that the restriction to the set of finite dimensional bimodules is generally a weaker condition than the original definition of Property T by Bekka.

Proposition 6. *Let X be a compact Hausdorff space and let $\mathcal{F}$ be the set of finite dimensional Hilbert bimodules of $C(X)$. Then $C(X)$ has Property $(T, \mathcal{F})$.*

Proof. Let $\mathcal{H}$ be a finite dimensional Hilbert bimodule of $C(X)$. Then $\mathcal{H} = L^2(X \times X, \mu)$, where μ has finite support and

$$(f\xi)(x, y) = f(x)\xi(x, y)$$

$$(\xi f)(x, y) = \xi(x, y)f(y)$$

for all $f \in C(X)$ and $\xi \in L^2(X \times X, \mu)$. Suppose that $\{\xi_i\}$ is a sequence of almost central vectors in $L^2(X \times X, \mu)$. Let $(x, y) \in X \times X$ such that $\mu(x, y) \neq 0$. If $x \neq y$, then there is $g \in C(X)$ such that $g(x) = 1$ and $g(y) = 0$. Since μ has finite support $|g(x)\xi_i(x, y) - \xi_i(x, y)g(y)| \to 0$. Then $\xi_i(x, y) \to 0$. It follows that there is a point $(x_0, x_0) \in X \times X$ such that $\mu(x_0, x_0) \neq 0$. Then the characteristic function $\xi_0 = \chi_{(x_0,x_0)}$ is a nonzero central vector. □

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